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Puzzles for Matrix Models of Chiral Field Theories

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Abstract: We summarize the field-theory/matrix model correspondence for a chiral $\mathcal{N}=1$ model with matter in the adjoint, antisymmetric and conjugate symmetric representations as well as eight fundamentals to cancel the chiral anomaly. The associated holomorphic matrix model is consistent only for two fundamental fields, which requires a modification of the original Dijkgraaf-Vafa conjecture. The modified correspondence holds in spite of this mismatch.

1 Introduction

A new method for determining the effective gaugino superpotential of confining $\mathcal{N}=1$ gauge theories was introduced by Dijkgraaf and Vafa [1]. Based on string theory considerations, they proposed to take the tree-level superpotential as the action of a dual holomorphic matrix model [2]. The conjecture was proved for a few nontrivial examples, via two distinct methods. One approach [3] uses covariant superfield techniques in perturbation theory to integrate out massive matter fields in a gaugino background. A different method was proposed in [4, 5], where it was shown that the loop equations of the matrix model coincide formally with chiral ring relations induced by certain generalizations of the Konishi anomaly.

We will discuss a model with gauge group $U(N)$ and chiral matter content chosen to allow for a straightforward large N limit [6]. What we mean by this is that the number of chiral superfields is independent of the rank of the gauge group despite constraints from anomaly cancellation. The matter content is given by a field Φ in the adjoint representation, two fields A, S in the antisymmetric and conjugate symmetric two-tensor representations, and eight fields $Q_1 \ldots Q_8$ in the fundamental representation to cancel the chiral anomaly. Because of the chiral spectrum, mass terms are not allowed so the methods of [3] are not applicable. We chose a tree level superpotential:

$$W_{\rm tree} = \mathrm{tr}\,[W(\Phi) + S\Phi A] + \sum_{f=1}^{8} Q_f^T S Q_f \quad , \tag{1}$$

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where $W=\sum_j \frac{t_j}{j} z^j$ is a complex polynomial of degree $d+1$.

2 The Generalized Konishi Anomalies

Let us discuss the generalized Konishi anomalies for this model, following [4] and [5]. The structure of the tree level superpotential implies:

$$j\frac{\partial W_{eff}}{\partial t_j} = \langle \mathrm{tr}\,(\Phi^j)\rangle \quad . \tag{2}$$

The strategy is to extract a set of chiral ring relations which determine the generating function $T(z)=\langle \mathrm{tr}\,(\frac{1}{z-\Phi})\rangle$ of the correlators $\langle \mathrm{tr}\,(\Phi^j)\rangle$. Then integration of (2) allows one to compute the effective superpotential up to a piece independent of t_j.

The generalized Konishi anomaly is the anomalous Ward identity for a local holomorphic transformation of the chiral superfields:

$$\mathcal{O}^{(r)} \longrightarrow \mathcal{O}^{(r)} + \delta\mathcal{O}^{(r)} \quad . \tag{3}$$

We will project onto the chiral ring, i.e. the equivalence class of operators which are annihilated by the anti-chiral supercharge $\bar{Q}_{\dot\alpha}$. As is well-known, these are in one-to-one correspondence with the lowest components of chiral superfields. The chiral ring relation induced by the generalized Konishi anomaly is:

$$\delta\mathcal{O}_I \frac{\partial W}{\partial \mathcal{O}_I} \equiv -\frac{1}{32\pi^2}\mathcal{W}_I^{\alpha\ J}\mathcal{W}_{\alpha,J}{}^{K}\frac{\partial(\delta\mathcal{O}_K)}{\partial\mathcal{O}_I} \quad , \tag{4}$$

where the capital indices enumerate a basis of the representation r. We will investigate the generalized Konishi relations corresponding to the field transformations:

$$\delta\Phi = \frac{\mathcal{W}^\alpha\mathcal{W}_\alpha}{z-\Phi} \quad , \quad \delta\Phi = \frac{1}{z-\Phi} \quad , \tag{5}$$

$$\delta A = \frac{\mathcal{W}^\alpha}{z-\Phi}A\frac{(\mathcal{W}_\alpha)^T}{z-\Phi^T} \quad , \quad \delta A = \frac{1}{z-\Phi}A\frac{1}{z-\Phi^T} \quad , \tag{6}$$

$$\delta S = \frac{1}{z-\Phi^T}S\frac{1}{z-\Phi} \quad , \quad \delta Q_f = \sum_{g=1}^{N_F}\frac{\lambda_{fg}}{z-\Phi}Q_g \quad . \tag{7}$$

In the last equation, λ is an arbitrary matrix in flavor space.

Writing $\mathcal{W}^2=\mathcal{W}^\alpha\mathcal{W}_\alpha$, we define:

$$R(z) := -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{\mathcal{W}^2}{z-\Phi}\right)\right\rangle \quad , \quad T(z) := \left\langle \mathrm{tr}\left(\frac{1}{z-\Phi}\right)\right\rangle \quad , \tag{8}$$

$$M(z) := \left\langle \mathrm{tr}\left(S\frac{1}{z-\Phi}A\right)\right\rangle \quad , \quad M_Q(z) := \sum_f\left\langle Q_f^T\frac{1}{z-\Phi^T}S\frac{1}{z-\Phi}Q_f\right\rangle \quad , \tag{9}$$

$$K(z) := -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(S\frac{\mathcal{W}^2}{z-\Phi}A\right)\right\rangle \quad , \quad L(z) := \sum_f\left\langle Q_f^T S\frac{1}{z-\Phi}Q_f\right\rangle \quad , \tag{10}$$

and introduce the degree $d-1$ polynomials:

$$f(z) \quad := \quad -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\mathcal{W}^2\right)\right\rangle \quad , \tag{11}$$

$$c(z) \quad := \quad \left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle \quad . \tag{12}$$

Assuming that vevs of spinor fields vanish due to Lorentz invariance, we find the following Ward identities for the generating functions (8-10):

$$
\begin{aligned}
R(z)^2 - 2W'(z)R(z) + 2f(z) &= 0 \ , && (13)\\
T(z)R(z) - W'(z)T(z) - 2R'(z) + c(z) &= 0 \ , && (14)\\
K(z) - \frac{1}{2}R(z)^2 &= 0 \ , && (15)\\
M(z) - R(z)T(z) - 2R'(z) &= 0 \ , && (16)\\
M_Q(z) + M(z) - R(z)T(z) + 2R'(z) &= 0 \ , && (17)\\
2L(z) - N_F R(z) &= 0 \ . && (18)
\end{aligned}
$$

Given a solution $(R(z), T(z))$ of these constraints, the quantities K, M, M_Q and L are determined by $R(z)$ and $T(z)$. Hence all solutions are parameterized by the $2d$ complex coefficients of the polynomials $f(z)$ and $c(z)$.

The generalized Konishi relations involving the flavors Q_f have an interesting implication. Expanding equations (15) and (18) to leading order in $1/z$ gives:

$$
S = \frac{1}{4}\sum_f \langle Q_f^T S Q_f\rangle = \frac{2}{N_F}\sum_f \langle Q_f^T S Q_f\rangle \ , \qquad (19)
$$

where $S = -\frac{1}{32\pi^2}\langle \mathrm{tr}\, \mathcal{W}^2\rangle$ is the gaugino condensate. If S is non-vanishing, then compatibility of these two equalities requires that we set $N_F = 8$, which is also necessary to cancel the chiral anomaly. Any other value is incompatible with the existence of a gaugino condensate.

3 The Matrix Model

The general conjecture of [1] suggests that the effective superpotential of our field theory should be described by the holomorphic [2] matrix model:

$$
Z = \frac{1}{|G|}\int_\Gamma d\mu e^{-\frac{\hat{N}}{g}\mathcal{S}_{mm}(\hat\Phi,\hat A,\hat S,\hat Q)} \ , \qquad (20)
$$

where $|G|$ is a normalization factor, Γ denotes a gauge equivalence class of paths in the complex matrix configurations space $\mathcal{M}$ with $\dim_{\mathbb{R}}(\Gamma) = \dim_{\mathbb{C}}(\mathcal{M})$ and the matrix model action is given by:

$$
\mathcal{S}_{mm}(\hat\Phi,\hat A,\hat S,\hat Q) = \mathrm{tr}\, \left[W(\hat\Phi) + \hat S\hat\Phi\hat A\right] + \sum_{f=1}^{\hat N_F}\hat Q_f^T\hat S\hat Q_f = \mathrm{tr}\, \left[W(\hat\Phi) + \hat\Phi\hat A\hat S + \sum_{f=1}^{\hat N_F}\hat Q_f\hat Q_f^T\hat S\right] \ . \qquad (21)
$$

We use the convention that all matrix model quantities are denoted by hatted capital letters. The measure

$$
d\mu = \bigwedge_{i,j}^{\hat N} d\hat\Phi_{ij} \bigwedge_{i<j=1}^{\hat N} d\hat A_{ij} \bigwedge_{i\le j=1}^{\hat N} d\hat S_{ij} \bigwedge_{f=1}^{\hat N_f}\bigwedge_{i=1}^{\hat N} d\hat Q_{i,f}
$$

is a top holomorphic form on the space of matrices. Since anomaly cancellation in our field theory requires $N_F = 8$, one is tempted to set $\hat N_F = 8$ as well. Note that we are

forced to use a purely holomorphic formulation as in [2], since one cannot impose a reality condition on the matrices (such as hermiticity) without breaking the gauge invariance of the matrix model.

It turns out that the naive identification (20) cannot hold in our case. The matrix model action is invariant under the (complexified) gauge group $GL(\hat{N}, \mathbb{C})$ acting as:

$$\hat{\Phi} \to U\hat{\Phi}U^{-1} \quad , \quad \hat{A} \to U\hat{A}U^T \quad , \quad \hat{S} \to (U^{-1})^T\hat{S}U^{-1} \quad , \quad \hat{Q}_f \to U\hat{Q}_f \, . \tag{22}$$

However, the measure $d\mu$ is *not* invariant under the central $\mathbb{C}^*$ subgroup of $GL(\hat{N}, \mathbb{C})$ unless $\hat{N}_F = 2$. Taking $U = \xi \mathbf{1}_{\hat{N}}$ with $\xi \in \mathbb{C}^*$, we have:

$$\hat{A} \to \xi^2 \hat{A} \ , \ \ \hat{S} \to \xi^{-2}\hat{S} \ \text{ and } \ \hat{Q}_f \to \xi\hat{Q}_f \ \ , \tag{23}$$

which gives:

$$Z = \xi^{\hat{N}(\hat{N}_F - 2)} Z \ \ . \tag{24}$$

Thus Z must either vanish or equal complex infinity! This means that the matrix model predicted by a naive application of the conjecture of [1] is not well-defined.

That subtleties can arise when attempting to apply the conjecture of [1] to chiral field theories is not completely unexpected, since most derivations of this conjecture up to date have concentrated on real matter representations, which prevent the appearance of net chirality[1]. The phenomenon we just discussed shows that one must modify the original conjecture of [1] in order to adapt it to the chiral context.

We also note that the superpotential involves only operators which are singlets under the flavor group. Therefore one might still hope to find a formal map between the loop equations and the Konishi anomalies which also involves only flavor singlets. This leads us to consider the matrix model with $\hat{N}_F = 2$. Then both the action (21) and the integration measure are invariant under $GL(\hat{N}, \mathbb{C})$ transformations of the form (22), where U is now an arbitrary complex invertible matrix.

Let us have a look at the loop equations of the (20). Although the correlation functions are not well defined unless $\hat{N}_F = 2$, we shall consider formal expressions with an arbitrary value of $\hat{N}_F$. This will allow us to re-discover the constraint $\hat{N}_F = 2$ as a consistency condition between the loop equations, similar to the way in which we rediscovered the condition $N_F = 8$ by using the Konishi constraints of the field theory.

We consider the generalized the matrix model resolvents:

$$\omega(z) = \frac{g}{\hat{N}}\mathrm{tr}\left(\frac{1}{z - \hat{\Phi}}\right) \quad , \quad k(z) = \frac{g}{\hat{N}}\mathrm{tr}\left(\hat{S}\frac{1}{z - \hat{\Phi}}\hat{A}\right) \quad , \tag{25}$$

$$m_Q(z) = \hat{Q}_f^T \frac{1}{z - \hat{\Phi}^T}\hat{S}\frac{1}{z - \hat{\Phi}}\hat{Q}_f \quad , \quad l(z) = \hat{Q}_f^T\hat{S}\frac{1}{z - \hat{\Phi}}\hat{Q}_f \ \ . \tag{26}$$

It is not hard to show that they fulfill the loop equations:

$$\begin{aligned}
\langle \omega(z)^2 - \frac{g}{\hat{N}}\omega'(z) - 2W'(z)\omega(z) + 2\tilde{f}(z) \rangle &= 0 \ , && (27)\\
\langle \frac{1}{2}\omega(z)^2 + \frac{1}{2}\frac{g}{\hat{N}}\omega'(z) - k(z) \rangle &= 0 \ , && (28)\\
\langle \omega'(z) + m_Q(z) \rangle &= 0 \ , && (29)\\
\langle \hat{N}_F\omega(z) - 2l(z) \rangle &= 0 \ , && (30)
\end{aligned}$$

[1]Note however [8] who study generalized Konishi anomalies for some chiral models, though without studying the associated matrix models.

where:

$$\tilde{f}(z) := \frac{g}{\hat{N}} \mathrm{tr} \frac{W'(z) - W'(\hat{\Phi})}{z - \hat{\Phi}} \tag{31}$$

is a polynomial of degree $d-1$. Note that the only dynamical input is represented by the polynomial $\tilde{f}(z)$.

The leading order in the large z expansion of the last two equations gives:

$$g = \langle \hat{Q}_f^T \hat{S} \hat{Q}_f \rangle = \frac{2}{\hat{N}_F} \langle \hat{Q}_f^T \hat{S} \hat{Q}_f \rangle \,. \tag{32}$$

Since we of course take $g \neq 0$, equations (32) are consistent only if $\hat{N}_F = 2$.

4 Relation between the matrix model and field theory

Consider the large $\hat{N}$ expansion of a generic matrix model expectation value $\langle \hat{X} \rangle$:

$$\langle \hat{X} \rangle = \sum_{j=0}^{\infty} \left(\frac{g}{\hat{N}} \right)^j \hat{X}_j \quad . \tag{33}$$

Applying this expansion to (27)-(30) it is clear that to leading order we obtain equations which are formally identical to the Konishi constraints for $R(z), K(z), M_Q(z)$ and $L(z)/4$. This implies of course that we identify the polynomials $f(z)$ and $\tilde{f}(z)$.

Taking into account also the terms at order $g/\hat{N}$ and introducing the operator $\delta := \sum_k N_k \frac{\partial}{\partial S_k}$, it is easy to see that $\delta\omega_0(z) + 4\omega_1(z)$ can be identified with $T(z)$ and $\delta k_0(z) + 4k_1(z)$ with $M(z)$.

To summarize, we recover the Konishi constraints from the large $\hat{N}$ expansion of the matrix model loop equations by making the following identifications:

Matrix Model	Field Theory
$\omega_0(z)$	$R(z)$
$\delta\omega_0(z) + 4\omega_1(z)$	$T(z)$
$\delta k_0(z) + 4k_1(z)$	$M(z)$
$k_0(z)$	$K(z)$
$4m_{Q0}(z)$	$M_Q(z)$
$4l_0(z)$	$L(z)$
$\tilde{f}_0(z)$	$f(z)$
$\delta\tilde{f}_0(z) + 4\tilde{f}_1(z)$	$c(z)$

This also implies that the effective superpotential is given by

$$W_{eff}(t,S) = \sum_i N_i \frac{\partial F_0}{\partial S_i} + 4F_1 + \psi(S) \quad . \tag{34}$$

Here $F = \log Z$ is the matrix model's partition function and ψ is a function which depends on S_i but not on the coefficients of W. We take $\psi(S) = \alpha \sum_{i=1}^{d} S_i$, with $\alpha = (N-4)\ln\Lambda$, where Λ is the scale of our field theory. Then one can check that $\psi(S)$ together with the non-perturbative contribution to F due to the measure of the matrix model give the Veneziano-Yankielowicz part of the effective superpotential [6].

5 Conclusions

Rather surprisingly, we found that the number of fundamental flavors N_F in field theory must be taken to differ from the number of flavors $\hat{N}_F$ in the dual matrix model. Nevertheless, there exists an exact if formal one-to-one map between the large $\hat{N}$ expansion of the loop equations of the matrix model and the Konishi anomaly constraints of the field theory. A better understanding of this mismatch between the numbers of flavors might be obtained by studying the string theory realization of this model. The basic analysis was performed in [7], where we showed that, unexpectedly, the geometric background involves a $\mathbb{Z}_4$ orientifold of an A_2 fibration and that the flavors correspond to eight fractional D5-branes.

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